Substituting (2.14) into (2.10)–(2.13), we obtain the following relationships:

$$\nabla \cdot (\varepsilon \vec{\mathbf{E}}_c) = 0, \tag{2.16}$$

$$\nabla \times \vec{\mathbf{E}}_c = i\omega\mu \vec{\mathbf{H}}_c, \tag{2.17}$$

$$\nabla \cdot \vec{\mathbf{H}}_c = 0, \tag{2.18}$$

$$\nabla \times \vec{\mathbf{H}}_c = -i\omega\varepsilon \vec{\mathbf{E}}_c, \tag{2.19}$$

where the complex permittivity of the medium is defined as

$$\varepsilon = \varepsilon_0(1+\chi) + i\frac{\sigma}{\omega}. \tag{2.20}$$

Except for ε_0, which is a fundamental, real-valued physical constant, all other parameters depend on the medium under consideration as well as on the frequency ω.

Problem 2.4: Verify that (2.16)–(2.19) follow from (2.3)–(2.9) and (2.14).

Solution for a Plane Wave

Any harmonic electromagnetic field satisfying the above equations is physically realizable. However, we will restrict our attention to solutions describing a *plane wave*. Such solutions have the form

$$\vec{\mathbf{E}}_c = \vec{\mathbf{E}}_0 \exp(i\vec{\mathbf{k}} \cdot \vec{\mathbf{x}} - i\omega t), \qquad \vec{\mathbf{H}}_c = \vec{\mathbf{H}}_0 \exp(i\vec{\mathbf{k}} \cdot \vec{\mathbf{x}} - i\omega t), \tag{2.21}$$

where $\vec{\mathbf{E}}_0$ and $\vec{\mathbf{H}}_0$ are constant (complex) vectors, and $\vec{\mathbf{k}} = \vec{\mathbf{k}}' + i\vec{\mathbf{k}}''$ is a complex *wave vector*. Thus

$$\vec{\mathbf{E}}_c = \vec{\mathbf{E}}_0 \exp(-\vec{\mathbf{k}}'' \cdot \vec{\mathbf{x}}) \exp[i(\vec{\mathbf{k}}' \cdot \vec{\mathbf{x}} - \omega t)], \tag{2.22}$$

$$\vec{\mathbf{H}}_c = \vec{\mathbf{H}}_0 \exp(-\vec{\mathbf{k}}'' \cdot \vec{\mathbf{x}}) \exp[i(\vec{\mathbf{k}}' \cdot \vec{\mathbf{x}} - \omega t)]. \tag{2.23}$$

These relationships imply that the vector $\vec{\mathbf{k}}'$ is normal to planes of constant phase (and thus indicates the direction of propagation of the wave crests), while $\vec{\mathbf{k}}''$ is normal to planes of constant amplitude. The two are not necessarily parallel. When they are, or when $\vec{\mathbf{k}}''$ is zero, the wave is called *homogeneous.*

The term $\vec{\mathbf{E}}_0 \exp(-\vec{\mathbf{k}}'' \cdot \vec{\mathbf{x}})$ gives the *amplitude* of the electric wave at location $\vec{\mathbf{x}}$. If $\vec{\mathbf{k}}''$ is zero, then the medium is nonabsorbing, because the amplitude is constant. The quantity $\phi \equiv \vec{\mathbf{k}}' \cdot \vec{\mathbf{x}} - \omega t$ gives the *phase*. The *phase speed* of the wave is given by

$$v = \frac{\omega}{|\vec{\mathbf{k}}'|} \tag{2.24}$$

Substituting (2.21) into (2.16)–(2.19) yields

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{E}}_0 = 0, \tag{2.25}$$

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{H}}_0 = 0, \tag{2.26}$$

$$\vec{\mathbf{k}} \times \vec{\mathbf{E}}_0 = \omega\mu\vec{\mathbf{H}}_0, \tag{2.27}$$

$$\vec{\mathbf{k}} \times \vec{\mathbf{H}}_0 = -\omega\varepsilon\vec{\mathbf{E}}_0. \tag{2.28}$$

Problem 2.5: Verify that (2.25) follows from (2.21) and (2.16) by (a) expanding (2.21) in terms of the individual components of the vectors $\vec{\mathbf{E}}_0$, $\vec{\mathbf{k}}$, $\vec{\mathbf{x}}$ and (b) substituting this expression into (2.16) and applying the divergence operator ($\nabla\cdot$). The remaining equations (2.26)–(2.28) are derived in an analogous way.

If we now take the vector product of $\vec{\mathbf{k}}$ with both sides of (2.27),

$$\vec{\mathbf{k}} \times (\vec{\mathbf{k}} \times \vec{\mathbf{E}}_0) = \omega\mu\vec{\mathbf{k}} \times \vec{\mathbf{H}}_0 = -\varepsilon\mu\omega^2\vec{\mathbf{E}}_0, \tag{2.29}$$

and use the vector identity

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}), \tag{2.30}$$

we see from (2.25) that the first term on the right is zero, thus

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{k}} = \varepsilon\mu\omega^2. \tag{2.31}$$

In the case of a homogeneous wave, the above simplifies to

$$(|\vec{\mathbf{k}}'| + i|\vec{\mathbf{k}}''|)^2 = \varepsilon\mu\omega^2 \tag{2.32}$$

or

$$|\vec{\mathbf{k}}'| + i|\vec{\mathbf{k}}''| = \omega\sqrt{\varepsilon\mu}. \tag{2.33}$$

Phase Speed

In a vacuum, $\vec{\mathbf{k}}'' = 0$, the permittivity of free space $\varepsilon \equiv \varepsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}, and the magnetic permeability $\mu \equiv \mu_0 = 1.257 \times 10^{-6}$ N A^{-2}. From (2.24), we have the following phase speed in a vacuum:

$$c \equiv 1/\sqrt{\varepsilon_0\mu_0}. \tag{2.34}$$

If we substitute the above numerical values of ε_0 and μ_0 into this expression, we obtain the speed of light in a vacuum $c = 2.998 \times 10^8$ m s^{-1}.

In a nonvacuum, we can write

$$|\vec{\mathbf{k}}'| + i|\vec{\mathbf{k}}''| = \omega\sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}\sqrt{\varepsilon_0\mu_0} = \frac{\omega N}{c}, \tag{2.35}$$

where the complex *index of refraction* N is given by

$$\boxed{N \equiv \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} = \frac{c}{c'}\,,} \tag{2.36}$$

with $c' \equiv 1/\sqrt{\varepsilon\mu}$. If N happens to be pure real (i.e., if $\vec{\mathbf{k}}'' = 0$ and the medium is therefore nonabsorbing), then c' may be interpreted as the phase speed of the wave within the medium. Even in absorbing media, this is still a reasonable, though approximate, way to view c'.

For most physical media, $N > 1$, which implies a reduced speed of light relative to that in a vacuum. It is important to keep in mind that N is not only a property of a particular medium but also generally a strong function of frequency.

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