For this case $dt(s)/ds \equiv W(s) = 0$ for any $s \neq s_0$ and infinite for $s = s'$. In fact, we can write

$$W(s) = \delta(s - s')\ , \tag{8.20}$$

where $\delta(x)$ is the Dirac δ-function. If you have not heard of this function before, then let me quickly summarize its key properties:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{8.21}$$

In other words, $\delta(x)$ is a function that is zero everywhere except at $x = 0$, where it is an infinitely tall, infinitely narrow spike. Despite these unusual properties, the area under the curve is defined to be finite and equal to one. That is,

$$\int_{-\infty}^{x} \delta(x - x')\ dx' = \begin{cases} 0 & x < x' \\ 1 & x > x' \end{cases} \tag{8.22}$$

This is of course consistent with (8.19) and (8.20).

Now if you think about it a bit, you will realize that the product of $\delta(x - x')$ with $f(x)$ is just the same δ-function, but multiplied by the value of $f(x)$ evaluated exactly at x'. Thus, (8.22) implies

$$\int_{x_1}^{x_2} \delta(x - x') f(x')\ dx' = \begin{cases} f(x) & x_1 < x < x_2 \\ 0 & \text{otherwise} \end{cases} \tag{8.23}$$

Let us now look at what this implies for (8.18), given (8.20):

$$I(S) = I(s_0)t(s_0) + \int_{s_0}^{S} B(s)\delta(s - s')ds\ , \tag{8.24}$$

where we will take s_0 to be an arbitrary point below our surface at s', so that $s_0 < s' < S$. Since the medium in question is opaque by assumption, the transmittance $t(s_0) = 0$, and we're left with

$$I(S) = \int_{s_0}^{S} B(s)\delta(s - s')ds\ , \tag{8.25}$$

or, invoking (8.23),

$$I(S) = B(s')\ . \tag{8.26}$$

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We have just confirmed, in an admittedly roundabout way, what we already knew: The emitted intensity from a opaque blackbody is just the Planck function B evaluated at the surface of the blackbody. Why did we go to all this trouble? My purpose was simply to demonstrate that (8.17) is quite general (for a nonscattering and nonreflecting medium) and is valid even when the transmittance t is a discontinuous function of distance along a path.

8.2 Radiative Transfer in a Plane Parallel Atmosphere

Let us now adapt (8.17) to a plane-parallel atmosphere. We will start by considering the case that a sensor is located at the surface ($z = 0$), viewing downward emitted radiation from the atmosphere. The appropriate form of the radiative transfer equation is then

$$\boxed{I^{\downarrow}(0) = I^{\downarrow}(\infty)t^* + \int_0^{\infty} B(z)W^{\downarrow}(z)dz \; ,} \tag{8.27}$$

where $z = \infty$ represents an arbitrary point beyond the top of the atmosphere, and $t^* \equiv \exp(-\tau^*/\mu)$ is the transmittance from the surface to the top of the atmosphere. Recall that we are using $B(z)$ here as a shorthand for $B_\lambda[T(z)]$, where $T(z)$ is the atmospheric temperature profile. Because the transmittance $t(0,z)$ between the surface and altitude z decreases with increasing z, our weighting function $W^{\downarrow}(z)$ in this case is given by

$$\boxed{W^{\downarrow}(z) = -\frac{dt(0,z)}{dz} = \frac{\beta_{\mathrm{a}}(z)}{\mu} t(0,z) \; .} \tag{8.28}$$

Unless the sensor is pointing at an extraterrestrial source of radiation, such as the sun, $I^{\downarrow}(\infty) = 0$. In this case, the first term on the right vanishes, and the observed downward intensity is strictly a function of the atmospheric temperature and absorption profiles.

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Now let's consider a sensor above the top of the atmosphere looking down toward the surface. We then have

$$\boxed{I^{\uparrow}(\infty) = I^{\uparrow}(0)t^* + \int_0^{\infty} B(z)W^{\uparrow}(z)dz \,,} \tag{8.29}$$

where

$$\boxed{W^{\uparrow}(z) = \frac{dt(z,\infty)}{dz} = \frac{\beta_{\mathrm{a}}(z)}{\mu} t(z,\infty) \,.} \tag{8.30}$$

Note the strong similarity between (8.27) and (8.29). Both state that the radiant intensity emerging from the bottom or top of the atmosphere is the sum of two contributions: 1) the transmitted radiation entering the atmosphere from the opposite side, and 2) a weighted sum of the contributions of emission from each level z within the atmosphere.

8.2.1 The Emissivity of the Atmosphere

Consider the case that $T(z) = T_a$, where T_a is the temperature of an isothermal atmosphere. Then $B[T(z)] = B(T_a) = constant$, so that (8.27), together with (8.28), can be written

$$I^{\downarrow}(0) = I^{\downarrow}(\infty)t^* + B(T_a) \int_0^{\infty} -\frac{dt(0,z)}{dz}\, dz \,. \tag{8.31}$$

The integral reduces to $t(0,0) - t^* = 1 - t^*$, yielding

$$I^{\downarrow}(0) = I^{\downarrow}(\infty)t^* + B(T_a)[1 - t^*] \,. \tag{8.32}$$

By the same token, (8.29)and (8.30) can be manipulated to yield

$$I^{\uparrow}(\infty) = I^{\uparrow}(0)t^* + B(T_a)[1 - t^*] \,. \tag{8.33}$$

Recall that the absorptivity of a nonscattering layer is just one minus the transmittance, and that Kirchhoff's Law states that absorptivity equals emissivity. The interpretation of the above two equations is thus remarkably simple: *The total radiant intensity is just the sum of*