The square line is of course an extreme and unrealistic case, as real lines have wings that extend indefinitely outward from the line center. Nevertheless, this example is useful as a simple illustration of the effects of non-gray absorption within a band. More importantly, the square line model is as far from being gray as you can get, so it defines an upper bound on the radiative effects of non-grayness.

Specifically, we find that the band absorption for large mass path has a limiting value of $W_0 / \Delta\tilde{\nu}$. Since our square line has no wings, it can never absorb a larger fraction of the incident (white) radiation than the ratio of the line width to the width of the spectral band.

For any realistic line shape — i.e., for one with tails extending to infinity, we expect that the reduction in transmission with increasing mass path will be

1. slower than that predicted by Beer's Law for a gray medium, but

2. faster and more complete than that expected from a square line of the same width.

Lorentz Line

Let's now look at the absorption behavior of a realistic line, as described by the Lorentz shape

$$f(\tilde{\nu}) = \frac{\alpha_L}{\pi[(\tilde{\nu} - \tilde{\nu}_0)^2 + \alpha_L^2]} , \tag{10.27}$$

for which the equivalent width when integrated over the entire spectrum is

$$W = \int_{-\infty}^{\infty} \left[1 - \exp\left(\frac{-Su\alpha_L}{\pi[(\tilde{\nu} - \tilde{\nu}_0)^2 + \alpha_L^2]} \right) \right] d\tilde{\nu} . \tag{10.28}$$

We have extended the lower limit of integration in the above from zero to $-\infty$ for the sake of being able to obtain an analytic solution. To simplify the integral into a standard form that can be looked up in a table, we further define the nondimensional mass path

$$\tilde{u} \equiv \frac{Su}{2\pi\alpha_L} , \tag{10.29}$$

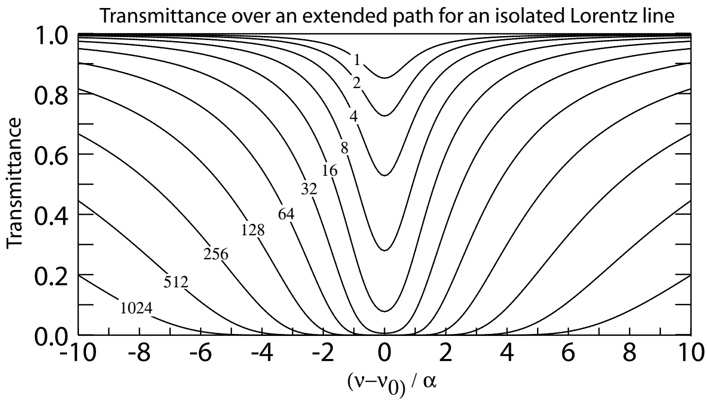

Fig. 10.1: The evolution of transmittance with increasing mass path for an isolated Lorentz line. For each curve, the line center optical path $\tilde{u}$ is equal to the label value divided by 2π.

which is nothing more than one-half of the optical path τ at the line center. The above integral can then be written

$$W = \int_{-\infty}^{\infty} \left[1 - \exp\left(\frac{-2\tilde{u}\alpha_L^2}{(\tilde{\nu}-\tilde{\nu}_0)^2 + \alpha_L^2} \right) \right] \, d\tilde{\nu} \,, \tag{10.30}$$

which has the solution

$$W = 2\pi\alpha_L L(\tilde{u}) \,, \tag{10.31}$$

where the *Ladenberg-Reiche function* $L(\tilde{u})$ can be expressed in terms of modified Bessel functions of the first kind of order 0 and 1:

$$L(\tilde{u}) = \tilde{u}e^{-\tilde{u}} \left[I_0(\tilde{u}) + I_1(\tilde{u}) \right] \,. \tag{10.32}$$

Admittedly, you won't find the modified Bessel functions on your pocket calculator, but apart from that minor inconvenience, they are standard mathematical functions. Here we will confine our attention to the behavior of $W(u)$ in the limit of very small and very large mass path. But wait: we already know what happens in the first case from (10.22), which is valid for *any* line shape — absorption in the weak line limit is simply proportional to u.

In the limit of large mass path (or, more precisely, large $\tilde{u}$), it can be shown that

$$W \approx 2\alpha_L\sqrt{2\pi\tilde{u}} = 2\sqrt{S\alpha_L u} \,. \tag{10.33}$$

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In other words, in the *strong line limit*, an isolated Lorentz line absorbs in proportion to the square root of the mass path.

The average transmission for a spectral band $\Delta\tilde{\nu}$ is, as usual,

$$\mathcal{T} = 1 - \frac{W}{\Delta\tilde{\nu}} , \tag{10.34}$$

where W is given by (10.31), *provided* that $\Delta\tilde{\nu}$ is large enough to encompass all significant absorption by the wings of the Lorentz line. If this condition is not satisfied, then the limits of integration employed in (10.28) are not applicable, and no closed-form solution exists. However, it is easy to convince yourself that the use of a narrower band $\Delta\tilde{\nu}$ will have the following consequences:

- The equivalent width W will be reduced relative to (10.31), because contributions from the far wings of the line are lost;

- The band-averaged transmittance $\mathcal{T}$ will be reduced, because the clearest portions of the spectrum furthest away from the line center are excluded.

The behavior of an isolated Lorentz line with varying $\tilde{u}$ is depicted graphically in Fig. 10.1. For the spectral interval depicted, the band transmittance is equal to the area under any given curve (labeled with values proportional to $\tilde{u}$), expressed as a fraction of the total area of the plot.

For small values of $\tilde{u}$, the transmission is close to unity everywhere, even at the line center. This represents the linear or weak line regime, for which $W = Su$. For large values ($\tilde{u} \gg 10$), the line becomes saturated in the center, so that those wavelengths can no longer contribute to further increases in $\mathcal{A}$ with increasing $\tilde{u}$. Instead, the width of the saturated zone increases, though at a slower rate than $\tilde{u}$ itself; this represents the strong line limit or square root regime (as noted above, this description is strictly valid only when the edges of the spectral window are far enough from the line center).

Unfortunately, we rarely have the luxury of dealing with isolated lines; rather, in the usual scenario we have dozens, if not hundreds of lines in our spectral interval. If the lines don't overlap (as might be the case at high altitudes), then their contributions to the band absorption $\mathcal{A}$ can be determined individually as for isolated