we can take them outside the integral. We get

$$I(\tau^*)e^{-\frac{\tau^*}{\mu}} - I(0) = \frac{-F_0\tilde{\omega}}{4\pi\mu\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)} p(\cos\Theta)\left[e^{\tau^*\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)} - 1\right] . \tag{11.30}$$

It may surprise you to learn that the above equation is valid both for downwelling radiation at the bottom of the atmosphere and for upwelling radiation at the top of the atmosphere. In the first case, we're interested in $I(0)$ for the case that $\mu > 0$:

$$I(0) = I(\tau^*)e^{-\frac{\tau^*}{\mu}} + \frac{F_0\tilde{\omega}}{4\pi\mu\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)} p(\cos\Theta)\left[e^{\tau^*\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)} - 1\right] . \tag{11.31}$$

In the second case, we want $I(\tau^*)$ for $\mu < 0$, which requires only a slight rearrangement:

$$I(\tau^*) = I(0)e^{\frac{\tau^*}{\mu}} + \frac{F_0\tilde{\omega}}{4\pi\mu\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)} p(\cos\Theta)\left[e^{\frac{\tau^*}{\mu_0}} - e^{\frac{\tau^*}{\mu}}\right] . \tag{11.32}$$

To summarize, the above equations give scattered radiances at the top and bottom of the atmosphere (or a thin cloud layer) for the special case that all of the above are satisfied: (a) multiple scattering is negligible, (b) $\tilde{\omega}$ and $p(\cos\Theta)$ are constant, and (c) the sole external illumination is a parallel beam source such as the sun. *It's important to recall the requirement that either $\tilde{\omega} \ll 1$ and/or $\tau^* \ll 1$ in order for the first of these requirements to be satisfied.*

Let's take things a step further. First, we'll focus only on the scattered atmospheric contribution to the radiance and drop the term that describes the direct transmission of radiation from the opposite side of the atmosphere (we can always add it back, if we want it). Second, we'll assume that the reason why we can neglect multiple scattering is that $\tau^* \ll 1$, and we'll further assume that μ_0 and μ are *not* much smaller than one. Taking advantage of the fact that, for small x, $e^x \approx 1 + x$, we can then simplify our equations to

$$\left.\begin{array}{ll} \text{For } \mu > 0, & I(0) \\ \text{For } \mu < 0, & I(\tau^*) \end{array}\right\} = \frac{F_0\tilde{\omega}\tau^*}{4\pi\mu} p(\cos\Theta) . \tag{11.33}$$

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The interpretation of the above equation is straightforward — so straightforward in fact, that we probably could have guessed it without going through all the previous steps. First, the quantity $F_0\tau^*$ tells us the magnitude of the extinguished solar flux (recall that this is valid only in the limit of small τ^*). Second, the quantity $(\tilde{\omega}/4\pi)p(\cos\Theta)$ tells us how much of the intercepted flux contributes to the scattering source term in a given new direction $\hat{\mathbf{\Omega}}$. Finally, the factor $1/\mu$ accounts (to first order) for the fact that you are looking through less atmosphere if you view it vertically than if you look toward the horizon; consequently the path-integrated contribution of scattering to the observed intensity increases toward the horizon.

Problem 11.3: If you have access to a decent plotting program, set things up so that you can conveniently plot $I(\tau^*)$ versus $-1 < \mu < 0$ using both (11.33) and (11.32) on the same graph. Assume no extraterrestrial source of radiation from direction $\mu \neq \mu_0$. Assume isotropic scattering. Determine the range of μ, μ_0, and τ^* for which the second equation is a good approximation to the first. When the two disagree significantly, describe the nature of the disagreement. Focus on values of $\tau^* \leq 0.1$, since we know that neither equation is valid unless the atmosphere is optically thin. Note also that (11.32) cannot be directly evaluated when $\mu_0 = \mu$, though it gives physically reasonable values in the limit as $\mu \to \mu_0$.

11.5 Applications to Meteorology, Climatology, and Remote Sensing

11.5.1 Intensity of Skylight

We imposed several seemingly drastic restrictions in deriving (11.33): $\tau^* \ll 1$, $\tilde{\omega}$ and $p(\cos\Theta)$ independent of τ, μ and μ_0 not too small. In fact, these assumptions are reasonably well justified for molecular scattering of visible and near-IR sunlight in the cloud- and haze-free atmosphere, as long as (a) you stay away from the blue and violet end of the spectrum, and (b) you don't get too close to the horizon.

Therefore, to evaluate the radiant intensity of the sky (apart from the direct rays of the sun itself) you need only specify the optical depth τ^* of the cloud-free atmosphere at the wavelength in question, supply a suitable phase function $p(\Theta)$, and substitute these into (11.33) for arbitrary μ and μ_0.

As will be shown in Chapter 12, the scattering phase function of air molecules in the visible band is

$$p(\Theta) = \frac{3}{4}(1 + \cos^2\Theta) . \qquad (11.34)$$

This so-called *Rayleigh* phase function is quite smooth and is perfectly symmetric with respect to forward and backward scattering ($g = 0$). The factor-of-two variation in intensity implied by the above phase function is relatively minor and is unlikely to be obvious to the eye, especially since it is such a smooth function of the scattering angle Θ. Consequently, we expect the radiant intensity of the sky to appear rather uniform, punctuated only by the narrow spike of high intensity associated with the directly transmitted light of the sun.

Although $p(\Theta)$ has the same shape for molecular scattering at all visible wavelengths, the optical depth τ^* of the cloud free atmosphere is a strong function of wavelength. In fact, it is shown in the next chapter that $\tau^* \propto \lambda^{-4}$. Thus, (11.33) implies that the intensity of skylight due to molecular scattering should also be proportional to λ^{-4} and, indeed, it is precisely this dependence that gives us the blue sky. It is also because τ^* stops being “small” at shorter wavelengths that we can’t trust (11.33) to give us accurate sky intensities in the blue and ultraviolet part of the spectrum.

Of course, even the cleanest air found in nature contains not only molecules, but other kinds of particles called aerosols. There are typically many thousands of aerosol particles in every cubic centimeter of air. Those of interest to us here have sizes ranging from 10^{-2} μm to $\sim$1 μm or larger. The scattering of visible light by such comparatively large particles (compared to molecules, that is!) is not as strongly dependent on wavelength as is molecular scattering; furthermore the scattering phase function for aerosols is not symmetric like the Rayleigh phase function but rather exhibits fairly strong forward scattering.