$\vec{\mathbf{p}}$. The physical dimensions of $\vec{\mathbf{p}}$ are charge times distance, which can be interpreted as the net amount of charge Q displaced times an effective displacement $\vec{\mathbf{x}}$.

For most particles of interest to us, the dipole moment of a small spherical particle is proportional to the strength of the external electric field:

$$\vec{\mathbf{p}} = \alpha \vec{\mathbf{E}}_0 \exp(i\omega t) , \tag{12.2}$$

where α is called the *polarizability* of the particle. It depends on the composition and the size of the particle, as well as on the frequency $\omega = 2\pi\nu$ of the incident wave. Note that α may be complex. Any nonzero imaginary part implies a phase difference between the real part of $\vec{\mathbf{p}}$ and the real part of $\vec{\mathbf{E}}$.

In summary, we have an oscillating dipole whose strength and orientation fluctuates in lockstep with the electric field due to the incident wave. But an oscillating dipole produces its own oscillating electric field, and these oscillations propagate outward at the speed of light. This is of course the origin of the scattered radiation.

Now imagine that the incident wave is traveling in direction $\hat{\mathbf{\Omega}}$ and you are positioned at a large distance $R \gg r$ from the dipole, in direction $\hat{\mathbf{\Omega}}'$. There are several facts we can jot down that will aid us in visualizing the relationship between the scattered wave at our location and the incident wave:

1. We know that in any EM wave, the electric field vector is perpendicular to the direction of propagation $\hat{\mathbf{\Omega}}$.

2. We are assuming here that $\vec{\mathbf{p}}$ is aligned with the electric field $\vec{\mathbf{E}}_0$ of the incident wave,[4] so $\vec{\mathbf{p}}$ is also perpendicular to $\hat{\mathbf{\Omega}}$.

3. Because of the symmetry of the charge distribution in the dipole, the electric field vector $\vec{\mathbf{E}}_{scat}$ of the scattered wave at any location must lie in the plane that contains both $\vec{\mathbf{p}}$ and $\hat{\mathbf{\Omega}}'$.

4. The *strength* of the electric field at your location is proportional to the *projection* of $\vec{\mathbf{p}}$ as seen from your direction. Specifically,

[4]In other words, we are assuming that the polarizability α is a *scalar* rather than a 3×3 *tensor* which would alter the direction of $\vec{\mathbf{p}}$ relative to $\vec{\mathbf{E}}_0$. This is always valid for spherical particles composed of an electrically isotropic substance like water.

Return to: A First Course in Atmospheric Radiation

$\vec{\mathbf{E}}_{\text{scat}}$ is zero if you are viewing the dipole "end on" and it is a maximum (for a given distance) when you are viewing it at right angles. We can put this in mathematical terms by saying that $\vec{\mathbf{E}}_{\text{scat}} \propto \sin\gamma$, where γ is the angle between $\vec{\mathbf{E}}_0$ (using Fact 2, above) and the scattered direction $\hat{\mathbf{\Omega}}'$.

5. Less obvious, but equally important, is that the power radiated by the dipole is proportional to the *acceleration* of the electric charge in the dipole. That is to say, a stationary dipole will create a static electric field but no propagating EM wave, and it will therefore radiate no energy. A vibrating dipole, on the other hand, induces a vibrating electric field (and therefore an outward propagating EM wave) whose amplitude is proportional to the *square* of the frequency of the vibration.

Facts 4 and 5 together, combined with (12.2), give us the following proportionality:

$$|\vec{\mathbf{E}}_{\text{scat}}| \propto \frac{\partial^2 \vec{\mathbf{p}}}{\partial t^2} \sin\gamma \propto \omega^2 \sin\gamma \ . \tag{12.3}$$

As discussed in section 2.5, the *power per unit area*, and therefore the intensity I, is proportional to the *square* of the electric field amplitude. Therefore, the scattered intensity is given by the following proportionality:

$$I \propto \omega^4 \sin^2\gamma \ . \tag{12.4}$$

We now want to recast the above proportionality in terms of the scattering angles Θ and Φ, where Θ is the angle between $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$, and Φ is the polar angle about $\hat{\mathbf{\Omega}}$ measured from an arbitrary starting point.

For convenience, we let the direction of incidence $\hat{\mathbf{\Omega}}$ coincide with the x-axis, and the incident electric field vector $\vec{\mathbf{E}}_0$ be aligned with the z-axis, consistent with Fact 2, above. We can then expand $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ in Cartesian coordinates as follows:

$$\hat{\mathbf{\Omega}} = (1, 0, 0) \ , \tag{12.5}$$

$$\hat{\mathbf{\Omega}}' = (\cos\Theta, \sin\Theta\sin\Phi, \sin\Theta\cos\Phi) \ . \tag{12.6}$$

Return to: A First Course in Atmospheric Radiation

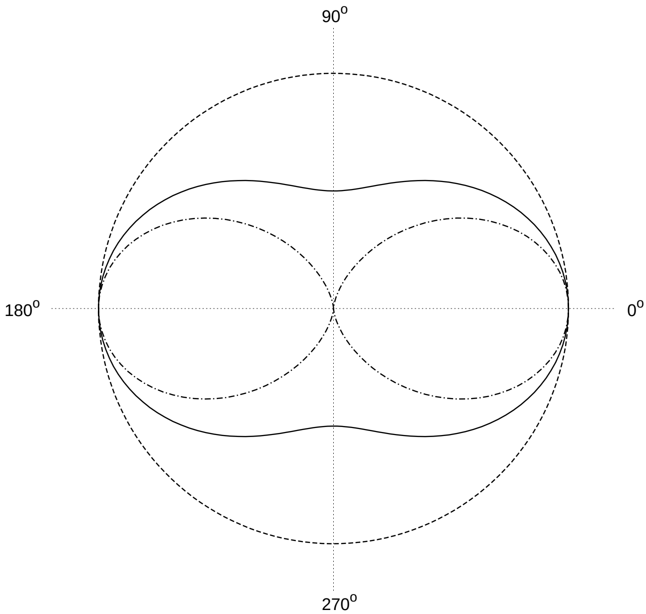

Fig. 12.2: Polar plot of the phase function for scattering by small particles (Rayleigh scattering). The outermost curve (dashed) represents the scattered intensity for directions $\hat{\mathbf{\Omega}}'$ lying in a plane *perpendicular* to the electric field vector of the incident wave. The innermost curve (dot-dashed) corresponds to directions lying in a plane *parallel* to the electric field vector. The solid curve represents the scattered intensity for unpolarized incident radiation, as given by (12.10).

This allows us to write

$$\begin{aligned}\cos\gamma &= \hat{\mathbf{z}}\cdot\hat{\mathbf{\Omega}}' \\ &= (0,0,1)\cdot\hat{\mathbf{\Omega}}' \\ &= \sin\Theta\cos\Phi\ ,\end{aligned} \tag{12.7}$$

and

$$\sin^2\gamma = 1-\cos^2\gamma \quad = 1-\sin^2\Theta\cos^2\Phi\ . \tag{12.8}$$

Substituting into (12.4) gives

$$I \propto \omega^4(1-\sin^2\Theta\cos^2\Phi)\ . \tag{12.9}$$

The above equation contains all of the essential features of what we will henceforth refer to as *Rayleigh scattering.* Before we continue, let's take a moment to interpret this result:

Return to: A First Course in Atmospheric Radiation