The result can be written as

$$\frac{1}{2}\frac{dI^{\uparrow}}{d\tau} = I^{\uparrow} - \tilde{\omega}(1-\overline{b})I^{\uparrow} - \tilde{\omega}\overline{b}I^{\downarrow} \; , \qquad (13.11)$$

or

$$\boxed{\frac{1}{2}\frac{dI^{\uparrow}}{d\tau} = (1-\tilde{\omega})I^{\uparrow} + \tilde{\omega}\overline{b}(I^{\uparrow} - I^{\downarrow}) \; ,} \qquad (13.12)$$

where

$$\overline{b} \equiv \int_0^1 b(\mu)\, d\mu \; . \qquad (13.13)$$

Repeating the above steps for the downward stream yields the analogous equation

$$\boxed{-\frac{1}{2}\frac{dI^{\downarrow}}{d\tau} = (1-\tilde{\omega})I^{\downarrow} - \tilde{\omega}\overline{b}(I^{\uparrow} - I^{\downarrow}) \; .} \qquad (13.14)$$

Equations (13.12) and (13.14) are the so-called *two-stream equations for diffuse incidence*.[2] Since $I^{\uparrow}(\tau)$ and $I^{\downarrow}(\tau)$ are unknown and appear in both equations, we're evidently dealing with a coupled pair of ordinary linear differential equations. The usual way to solve such a system is to combine them into a single second-order differential equation, apply boundary conditions, and solve for the specific boundary conditions of interest. But before we do, let's take a closer look at the mean backscatter fraction $\overline{b}$.

The Backscatter Fraction and g

The mean backscatter fraction $\overline{b}$ is explicitly related to the scattering phase function $p(\mu, \mu')$ via (13.8) and (13.13). The properties of the phase function can in turn be partly characterized via the asymmetry parameter g, which was defined by (11.20). It follows that there could be some kind of systematic relationship between $\overline{b}$ and g that would allow us to replace $\overline{b}$ in (13.12) and (13.14) with a suitable

[2] With relatively little effort, one can generalize the above equations to accommodate illumination at the top of the atmosphere by a direct beam of radiation from the sun — see for example TS02, Ch. 6.

function of g. This possibility can be made clearer by considering three special cases.

If scattering is perfectly isotropic [$p(\cos\Theta) = p(\mu, \mu') = 1$], then $g = 0$. In this case, regardless of the direction from which the radiation comes originally, it is equally likely to be scattered into either hemisphere, so clearly $\bar{b} = 1/2$.

If $g = 1$, this implies that all radiation is scattered in exactly the same direction as it was traveling before being scattered. Thus, absolutely no radiation can ever be scattered back into the opposite hemisphere; therefore $\bar{b} = 0$ for this case. Likewise, if $g = -1$, then all radiation is scattered into the opposite hemisphere, and $\bar{b} = 1$.

To summarize, we have the following known mappings between g and $\bar{b}$:

$$g = -1 \quad \rightarrow \quad \bar{b} = 1$$

$$g = 0 \quad \rightarrow \quad \bar{b} = \frac{1}{2}$$

$$g = 1 \quad \rightarrow \quad \bar{b} = 0$$

If we now *assume* that the relationship between g and $\bar{b}$ is linear,[3] then we can write

$$\bar{b} = \frac{1-g}{2} \,. \tag{13.15}$$

Making this substitution in (13.12) and (13.14), we have

$$\boxed{\frac{1}{2}\frac{dI^{\uparrow}}{d\tau} = (1-\tilde{\omega})I^{\uparrow} + \frac{\tilde{\omega}(1-g)}{2}(I^{\uparrow} - I^{\downarrow}) \,,} \tag{13.16}$$

$$\boxed{-\frac{1}{2}\frac{dI^{\downarrow}}{d\tau} = (1-\tilde{\omega})I^{\downarrow} - \frac{\tilde{\omega}(1-g)}{2}(I^{\uparrow} - I^{\downarrow}) \,.} \tag{13.17}$$

[3]This is another approximation; see TS02 Section 7.5 for a full discussion of the relationship between $\bar{b}$ and g.

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13.2.3 Solution

We start by adding and subtracting (13.16) and (13.17) to obtain

$$\frac{1}{2}\frac{d}{d\tau}(I^{\uparrow} - I^{\downarrow}) = (1 - \tilde{\omega})(I^{\uparrow} + I^{\downarrow}) , \tag{13.18}$$

$$\frac{1}{2}\frac{d}{d\tau}(I^{\uparrow} + I^{\downarrow}) = (1 - \tilde{\omega} g)(I^{\uparrow} - I^{\downarrow}) . \tag{13.19}$$

We then differentiate (13.19) to get

$$\frac{d^2}{d\tau^2}(I^{\uparrow} + I^{\downarrow}) = 2(1 - \tilde{\omega} g)\frac{d}{d\tau}(I^{\uparrow} - I^{\downarrow}) . \tag{13.20}$$

But note now that the derivative on the right hand side can be replaced with an expression obtained from (13.18), yielding

$$\frac{d^2}{d\tau^2}(I^{\uparrow} + I^{\downarrow}) = 4(1 - \tilde{\omega} g)(1 - \tilde{\omega})(I^{\uparrow} + I^{\downarrow}) . \tag{13.21}$$

Applying the same procedure as above to (13.18) gives

$$\frac{d^2}{d\tau^2}(I^{\uparrow} - I^{\downarrow}) = 4(1 - \tilde{\omega} g)(1 - \tilde{\omega})(I^{\uparrow} - I^{\downarrow}) . \tag{13.22}$$

These two equations are the same, except that in the first one the independent variable is $I^{\uparrow} + I^{\downarrow}$ while in the second it's $I^{\uparrow} - I^{\downarrow}$. We can therefore kill two birds with one stone by solving the single equation

$$\frac{d^2 y}{d\tau^2} = \Gamma^2 y , \tag{13.23}$$

where

$$y \equiv (I^{\uparrow} + I^{\downarrow}) \quad \text{or} \quad y \equiv (I^{\uparrow} - I^{\downarrow}) , \tag{13.24}$$

and

$$\boxed{\Gamma \equiv 2\sqrt{1 - \tilde{\omega}}\sqrt{1 - \tilde{\omega} g} .} \tag{13.25}$$

The general solution is

$$y = \alpha e^{\Gamma\tau} + \beta e^{-\Gamma\tau} . \tag{13.26}$$