APPENDIX A

Representing the Phase Function

A.1 Legendre Polynomial Expansion

Except in the Rayleigh limit, real particles always have phase functions that can't be expressed exactly using a closed-form mathematical expression (see Chapter 12). Therefore you will *almost always* have to employ some kind of approximate representation. Your approximate representation must allow the shape of $p(\cos\Theta)$ to be encoded to whatever level of accuracy is required for your application. Often, this means expressing $p(\cos\Theta)$ as an infinite series of suitable orthogonal basis functions — in this case, the Legendre polynomials $P_l(\cos\Theta)$:

$$p(\cos\Theta) = \sum_{l=0}^{\infty} \beta_l P_l(\cos\Theta) \ . \tag{A.1}$$

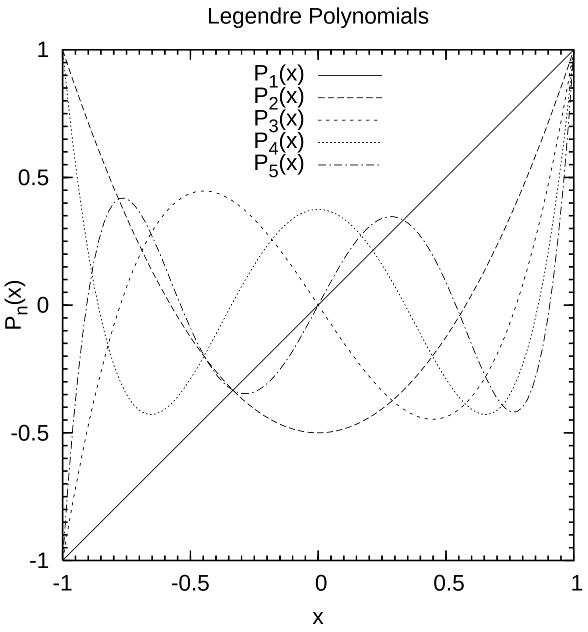

Fig. A.1: Examples of Legendre polynomials.

The first few Legendre polynomials are given by

$$P_0(x) = 1 \tag{A.2a}$$
$$P_1(x) = x \tag{A.2b}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{A.2c}$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{A.2d}$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{A.2e}$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \tag{A.2f}$$

and they satisfy the orthogonality condition

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0, & n \neq m \\ \dfrac{2}{2n+1}, & n = m \end{cases} \tag{A.3}$$

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