size of its shadow which is in turn equal to its physical cross-section, and its extinction efficiency Q_e is therefore equal to one. Since the bowling ball is black, we can further assume that its single scatter albedo is within a percent or two of zero. Finally, since it is opaque, much of the radiation that *is* reflected from its surface will be directed more or less backwards, so we can anticipate an asymmetry parameter $g \ll 1$.[1]

The same bowling ball examined from the point of view of the exact Mie theory (remember that geometric optics is an approximation!) looks quite different. In the limit of large x, we know that Mie theory predicts $Q_e = 2$. In other words, the bowling ball apparently extinguishes *twice* as much radiation as might be inferred from the size of its shadow alone! Earlier, we referred to this apparent discrepancy as the *extinction paradox*. Also, the Mie theory-derived single scatter albedo will be at least one-half, and the asymmetry parameter will be close to one! These values are all greatly at odds with our intuition, yet they are correct according to the exact theory.

"Common sense"	Exact Theory
$Q_e \approx 1$	$Q_e \approx 2$
$\tilde{\omega} \approx 0.01$	$\tilde{\omega} \approx 0.5$
$g \approx 0$	$g \approx 1$

This admittedly extreme example is one in which the intuitive (approximate) properties are actually more germaine than the exact ones. Why? Because in the exact results for the bowling ball, fully half of the total radiation extinguished is extinguished *only in a very narrow, literal sense of the word.* That half consists of radiation passing *near* the bowling ball that is deflected by an almost infinitesimal amount from its original direction. Technically speaking, it *is* scattered. And of course that infinitesimally deflected radiation factors into the computed values of $\tilde{\omega}$ and g. But for *any practical computational purpose*, it is a red herring, and we'd be better off using the much different, and less "correct," geometric optics results for our bowling ball.

[1]For a perfectly reflecting smooth sphere, a geometric optics calculation yields an asymmetry parameter of exactly zero. For an absorbing sphere with specular reflecting surface obeying the Fresnel relations, forward scattering will be slightly favored owing to the higher reflectivity for glancing rays.

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To summarize: Capturing the precise shape and intensity of the forward diffraction peak for a bowling ball would require the inclusion of extremely large number of terms in the Legendre polynomial expansion of the phase function. The forward diffraction peak is not nearly as narrow for cloud droplets as it is for bowling balls, but the same basic problem (in milder form) exists for solar radiative transfer in clouds. The computational effort needed to account for the forward peak in the phase function is not only unwelcome, it is also utterly pointless unless for some reason we really *have* to compute $I(\mu, \phi)$ with extremely high angular resolution.

The way to deal with this problem is to split the mathematical representation of the phase function into two pieces, one that accounts for the forward diffraction peak and another that captures the rest of the phase function. Specifically, we pretend that the diffraction peak can be adequately represented by a Dirac δ-function:

$$p(\cos\Theta) \approx A p'(\cos\Theta) + 4B\delta(\cos\Theta - 1) \; , \tag{A.6}$$

where $p'(\cos\Theta)$ is the *δ-scaled phase function*, and the coefficients A and B determine how the total phase function is partitioned between the two pieces. If we require $p'(\cos\Theta)$ to be properly normalized in its own right, then we can eliminate A as follows:

$$\frac{1}{2}\int_{-1}^{1}\left[A p'(\cos\Theta) + 4B\delta(\cos\Theta - 1)\right] d\cos\Theta = 1 \; , \tag{A.7}$$

$$A + 2B\int_{-1}^{1}\delta(\cos\Theta - 1) d\cos\Theta = 1 \; , \tag{A.8}$$

$$A + B = 1 \quad \rightarrow \quad A = 1 - B \; , \tag{A.9}$$

so that

$$p(\cos\Theta) \approx (1 - B)\, p'(\cos\Theta) + 4B\delta(\cos\Theta - 1) \; . \tag{A.10}$$

The asymmetry parameter g' of the scaled phase function $p'(\cos\Theta)$ should be chosen so as to be consistent with the asymmetry g of the original phase function:

$$\begin{aligned} g &\equiv \frac{1}{2}\int_{-1}^{1} x p(x)\, dx \\ &= \frac{1}{2}\int_{-1}^{1} x\left[(1-B)\, p'(x) + 4B\delta(x-1)\right]\, dx \\ &= (1-B)\,\frac{1}{2}\int_{-1}^{1} x p'(x)\, dx + 2B\int_{-1}^{1} x\delta(x-1)\, dx \\ &= (1-B)\, g' + B\,, \end{aligned} \tag{A.11}$$

so that the *scaled asymmetry parameter* is

$$\boxed{g' = \frac{\beta_1'}{3} = \frac{g-B}{1-B}\,.} \tag{A.12}$$

Note that by finding g', we automatically determine the second coefficient in the Legendre polynomial expansion of $p'(\cos\Theta)$. A similar matching procedure can be used to find subsequent coefficients β_l', given our choice of B and the expansion coefficients β_l for the original phase function.

Although the optimal choice of B is a bit ill-defined, the goal is to put as much of the diffraction peak as possible into the δ-term in (A.6) without doing violence to the rest of the phase function. Specifically, you want to be able to accurately represent the scaled phase function $p'(\cos\Theta)$ with significantly fewer terms in a Legendre polynomial expansion than would have been required for the original phase function $p(\cos\Theta)$.

Recall that the most general form of the radiative transfer equation was given by (11.9). If we neglect thermal emission, it becomes

$$\frac{dI(\hat{\mathbf{\Omega}})}{d\tau} = -I(\hat{\mathbf{\Omega}}) + \frac{\tilde{\omega}}{4\pi}\int_{4\pi} p(\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) I(\hat{\mathbf{\Omega}}')\, d\omega'\,. \tag{A.13}$$

If we substitute (A.10) into the above, we have

$$\begin{aligned} \frac{dI(\hat{\mathbf{\Omega}})}{d\tau} &= -I(\hat{\mathbf{\Omega}}) \\ &\quad + \frac{\tilde{\omega}}{4\pi}\int_{4\pi}\left[(1-B)\, p'(\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) + 4B\delta(\hat{\mathbf{\Omega}}'\cdot\hat{\mathbf{\Omega}} - 1)\right] I(\hat{\mathbf{\Omega}}')\, d\omega'\,, \end{aligned} \tag{A.14}$$

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