Excerpt from A First Course in Atmospheric Thermodynamics by G.W. Petty

APPENDIX B

Physical Problem Solving: A Tutorial

Homework sets requiring you to solve physical problems are an essential teaching tool in any upper-division physical science course. If you are like most students entering a junior-level course in atmospheric physics, you have had few opportunities yet to *apply* the concepts and techniques of calculus and physics to real-world problems like those encountered in meteorology.

It is easy to overlook the range of distinct learned skills that must be brought to bear in the solution of almost any problem. If you are still weak in any one or more of these skills (and who among us is fortunate enough to be *born* with any of them?), then the whole process will become an ordeal—an exercise in frustration rather than learning.

I have found that most students at this level have also not yet become accustomed to working with physical quantities *symbolically*, rather than numerically. And, sadly, many students (including the author) manage to get through their undergraduate years without developing a real appreciation for the power of *dimensional consistency* as a tool in the construction and debugging of problem solutions.

The purpose of this appendix is to serve as a brief tutorial on how experienced scientists and engineers attack physical problems.

I will highlight specific strategies that you would do well to adopt as habits not only for the problems you encounter in this course but in your future work as well. My insistence on symbolic rather than numerical problem-solving is a good example: my goal is not to make life more difficult but rather to make it *easier* in the long run! You will make fewer "dumb" mistakes, you will use your calculator less, your reasoning will be clearer, and—maybe most importantly, at this level at least—it will be easier for your instructor to spot what you did *right* in your homework and exam solutions! The only thing "hard" about symbolic problem-solving is that you have probably spent much more time doing things the other way.

Before we examine the mechanics of the physical computations, let us first review the entire process of solving a problem that might be given to you in a homework set or on an exam.

B.1 Five Steps to Success

The solution of any homework or exam problem, regardless of the details of the problem, invariably entails (or should entail) the following steps, the first of which has ideally been completed before you even see the problem:

1. Knowledge—learning the relevant facts and theory.
2. Analysis—applying familiar facts to an unfamiliar problem.
3. Execution—the mechanics of working and verifying your solution.
4. Validation—convincing yourself that your answer is correct.
5. Understanding—drawing the appropriate lessons from the exercise.

If you struggle with any of the first three, then your success in obtaining a solution at all may be in doubt. If you don't bother with the fourth, then you won't know until you get your homework or exam back whether your solution was correct. Finally, if you don't follow through with the fifth step, then the effort will be at least partly wasted, even if your solution is correct.

Let us review each of these steps in turn.

B.1.1 Acquiring relevant knowledge

The primary purpose of your instructor's lectures and your textbook is to impart *knowledge* of the subject matter of your course. Knowledge includes (among other things) *observed facts* (e.g., the composition of the atmosphere) and *governing principles* (e.g., the Ideal Gas Law). Problem sets are usually designed to give you practice *applying* the knowledge you have gained and, in some cases, to cultivate deeper insight into How Things Work.

It is not enough to just memorize a vast hoard of facts; you also must understand how they are interconnected. For example, knowledge of the composition of the atmosphere is essential for the computation of the relevant gas constant in the Ideal Gas Law. Knowledge of the characteristic motions in the atmosphere is what justifies our use of the Hydrostatic Relation.

In both your lectures and your reading, you must therefore pay close attention not only to the facts presented to you but also to the *context* in which they are presented:

- *Why* is the instructor or author making a particular point? Is it just knowledge for knowledge's sake, or is it an important brick in a larger wall?

- *What process or situation* does a particular equation describe? Knowing an equation by heart is of no value if you don't know what it is good for.

- *What assumptions* were used to obtain a particular equation? Even if that equation seems to describe something you need, such as the relationship between pressure and altitude, it is worse than useless if its underlying assumptions are invalid for your particular problem.

Actively answering *all* of the above questions in your own mind is an essential part of *studying.* If you passively read your textbook as if it were a mere recitation of disconnected facts, you will not be prepared to correctly *apply* your knowledge. The same warning applies if you read a homework problem first and then dive into your textbook on a scavenger hunt for the one key fact or formula needed to solve it.

B.1.2 Analyzing the problem

The next step is to work out a complete and continuous pathway from the knowledge at your disposal to the desired solution. Typically, your job is to select from your storehouse of facts some sturdy links that can be assembled into a continuous, logical chain leading from your givens to your solution. If any link is missing, then the chain is incomplete and therefore worthless – you will need to take stock and see if there's another piece of information that you overlooked. Sometimes that missing information is not a textbook formula or fact but rather a reasonable assumption, also known as "common sense." *Don't underestimate the power of common sense to help you hack your way through a vast thicket of facts and equations that might seem superficially relevant to your problem!*

If any step in the process takes you down a blind alley, you must recognize that and back up. Do not make unwarranted "leaps of faith" from that blind alley to your solution and hope that the grader won't notice!

Also, there are often both short and long routes to any given destination. When driving from Los Angeles to San Diego, it is theoretically possible (and technically not incorrect) to take a detour through New York, but I don't recommend it! The more complete your grasp of the "big picture", the more likely it is that you will recognize the shortest and simplest path. In my experience, it is very common for students to overlook a quick and easy pathway and pursue a complex, time-consuming one that leads to an identical answer, but with far greater investment and far greater risk of a mechanical breakdown.

Regardless of the path you take, you need to ensure that it is complete and valid. Every step must be logically correct, and every step should take you demonstrably closer to where you want to go. It may sometimes be helpful to write out (if only for your own benefit) what each step in the solution does for you —- what did you know when you started, what do you know now, and how does that get you closer to your desired solution? If you can't answer those questions (especially the last one), then you are not *solving*, you are *groping!*

A key test of the validity of your path is that it satisfies the requirement for *dimensional consistency*. Physical calculations inevitably involve not only numbers but also physical dimensions such as length, mass, time, etc. There are strict rules for mathematical operations on physical dimensions, and these rules can be powerful tools for uncovering errors in logic. The subject of physical dimensions is reviewed in greater detail in Section B.3.2. I urge you to review it carefully.

To summarize, you should *plan your solution* by answering the following questions for yourself:

1. What information are you being given as part of the problem?
2. What are you being asked to determine or calculate?
3. Which additional observed facts or reasonable assumptions are likely to be relevant?
4. Which governing principles or relationships apply?
5. Most importantly, how do the facts, assumptions, and relationships at your disposal allow you to logically get from your given information (1) to your desired solution (2)?

B.1.3 Executing the solution

Once you have mapped out your strategy for solving the problem, it is time for the implementation. You need to

1. Assign suitable symbols to represent each relevant quantity.
2. Specify values (including units) for those quantities, if applicable.
3. Manipulate the symbols algebraically to obtain the solution to your problem expressed *symbolically* in terms of your givens.
4. Verify the *dimensional consistency* of your solution.
5. Compute a *numerical solution* based on the specified values of the given variables.

All of the above steps are straightforward and can become second nature to any sufficiently motivated student regardless of native aptitude for science. Fluency is acquired not so much by *study* but rather by *practice*. Most of you, in fact, already *know* the relevant operations; it is sheer *repetition* that will make them automatic. There is, however, so much more to say about several of these steps that I have carved out a whole separate section (B.3) for that lofty purpose.

B.1.4 Validating your result

You have arrived at an answer and duly recorded it on your homework sheet. The dimensions work out—the problem asked for a radius, and sure enough, your answer has units of length.

At this point, many students consider their work done, put their pencil down, turn in their answer sheet, and, just maybe, cross their fingers. But it is not necessary to leave everything to chance. You can reduce the chance of turning in nonsense, and possibly getting zero credit[1] for your hard work, by taking a few extra minutes to think about whether your answer makes sense. Here are some examples:

- If your solution consists of a mathematical relationship, what variables appear in it? Are they the variables you expect based on your understanding of the problem? Is the relationship what you would expect it to be—for example, if x increases, should $y(x)$ increase or decrease based on your understanding of the role of x?

- When you substitute reasonable values for the variables in the relationship, can the result be positive, negative, or zero? Is the sign you get what you would expect it to be?

- What magnitude, or range of magnitudes, do you expect your quantity to take on in typical situations? Is the magnitude of your result consistent with the expected range? If the quantity is a vector, is the direction reasonable?

- Is there a special case or a limit for which you *know* what the answer should be (either exactly or approximately), and does your solution yield that answer?

[1]While every instructor has his or her own policies, my own is to give zero credit for an answer that is *obviously* wrong *unless* the student makes clear that they *recognize* that something is wrong!

B.1.5 Understanding your result

Many assigned problems aren't intended merely to test your knowledge or exercise a particular skill but also to drive home a specific point. What role does condensation and precipitation play in the occurrence of chinook winds? How large, really, is the effect of temperature variations on the change of pressure with height?

It follows that if you simply write up your problem solution, convince yourself that it appears to be mathematically and numerically correct, and then (mentally) walk away from it, you are shortchanging your own meteorological education. Your insight into the subject matter will mature more rapidly if you routinely set aside a little extra time to reflect on the following questions:

1. What general principle does the problem demonstrate?

2. To which parameters of the problem is the outcome most sensitive? Examine this question not only in mathematical terms, but also keeping in mind the likely range of parameter values encountered in the real atmosphere. For example, both the acceleration due to gravity g and the temperature T might appear in a particular relationship. While both vary, T at the Earth's surface is likely to vary by a far greater amount than g and therefore to have dominant influence on variations in your result.

3. Are there values for the parameters of your problem for which your solution "breaks," either mathematically (e.g., division by zero) or because some key assumption in its derivation is likely to be violated (e.g., $x \ll 1$)? Could those parameter values actually occur in the Earth's atmosphere? On another planet?

4. In addition to the specific scenario described in the problem, can you imagine other real-world scenarios to which the methods and/or general principles of the problem might apply?

B.2 The Solution Write-Up—Habits to *Unlearn*

B.2.1 Recipe for disaster

The best way to make the case for developing a disciplined and mature approach to your problem write-ups is to demonstrate the clear shortcomings of the alternative approach. The following pretty well captures what I often see on the first homework sets turned in by new students in my own undergraduate courses:

1. Use the numerical values of the given variables in the problem to compute a new, intermediate value. Write down that value with anywhere from 1 to 10 significant figures.

2. Ignore the units and assume that "it will all work out." Or, if unit conversions are required and are, in fact, explicitly worked out, let these appear as long, cumbersome chains of multiplications and divisions reminiscent of high school chemistry.

3. Use the above intermediate values to compute yet another intermediate value. The number of significant figures (i.e., precision) assigned to the new value may or may not have anything to do with the precision retained for the results of the previous steps.

4. Repeat above as necessary until you have a (purported) numerical value for the solution to the problem.

5. Convert to the desired final units. Or not! Or forget to even indicate what the units of the answer are.

It is quite possible, if you are reasonably careful, to obtain the correct answer to an assigned problem using the above procedure. Nevertheless, this *ad hoc, numerically oriented* approach to physical computations is both inefficient and error-prone. Let us look at some of the pitfalls:

- Every time you compute and write down an intermediate numerical value, you are necessarily rounding off the result. Depending on how much you round, the imprecision introduced each time can accumulate to give significant errors in your final answer.

- It can be almost impossible to spot either computational or logical errors when either you or the grader examines the mass of numbers and unit conversions scribbled on your page. Furthermore, the likelihood of a computational error (e.g., dropping a digit or punching a wrong button) increases each time you pick up your calculator.

- A lot of effort often goes into converting units as you go through each step. Much of this effort is wasted, and it increases the likelihood of errors.

- It obscures the underlying general physical relationships governing the variable you are solving for. Does variable x increase or decrease with increases in variable y? Does variable y even matter? If your approach is strictly numerical, you might not notice that y cancels out and is not even a factor in your final solution.

- If you need to repeat the calculation for the same problem but for a different set of parameter values, you have to repeat the entire procedure from scratch, again wasting time and increasing the likelihood of errors.

- It is difficult for an instructor to give partial credit for a problem solved in the above manner if the final answer turns out to be numerically incorrect because it can be impossible to tell whether the error was conceptual (the student didn't know what he/she was doing) or merely computational (the student dropped a sign somewhere).

A few of the hazards mentioned above are illustrated by the example on the next page, which is not at all unusual among the solutions I get from new students in my courses.

B.2.2 A bad example

Problem: Estimate the *total mass* (in kg) of the atmosphere M_{tot}, using the following information: Acceleration due to gravity g = 32.17 ft/s^2

Radius of the Earth R_E = 3960 miles
Average sea level pressure p_0 = 1013.2 millibars
Conversions: 1 m = 3.28 ft 1 mile = 1609.756 m

$$\frac{1013.2\ \text{mb} \times 100\ \frac{\text{Pa}}{\text{mb}} \times \frac{\text{N/m}^2}{\text{Pa}} \times \frac{\text{kg m/sec}^2}{\text{N}}}{32.17\ \text{ft/sec} \times 1\text{m}/3.28\text{ft}} =$$

$$\approx 10330.41965$$

$$4\pi(3960)^2\ \text{miles} = 197060797.4\ \text{miles} \times 1609.756\ \frac{\text{m}}{\text{mile}} \times 10330.41965$$

$$= 3.277013666 \times 10^{+15} \leftarrow \text{ANSWER (Mass of the atmosphere)}$$

The above solution is not only ugly, but anyone else reading it (e.g., your instructor) may have a hard time following the reasoning. Nevertheless, the *reasoning* is, in fact, correct, and the string of numerical calculations is *almost* correct in terms of finding the value of the total mass of the atmosphere. But the numerical result is wildly wrong, and it is due to exactly one fairly trivial error in the computation. But would you notice it if this were your own solution? Would you be able to spot the error if you were the grader and, if not, *would you even consider awarding partial credit?*

Problem B.1: a) Note the clock time as you start this exercise.

b) Find the computational error in the above solution. For reference, use the complete and correct solution found in Section B.3.6.

c) Record the total time it took you to find the error.

d) Multiply the above time by the number of problems on a typical homework set (assume ten unless otherwise instructed) and by the approximate number of students in your class. Express your answer in units of hours.

B.3 The Solution Write-Up—Habits to *Learn*

Let us now look at how to undertake physical calculations and write up problem solutions the way a professional would. The basic ideas can be summarized as follows:

- We begin our solution with *abstract* manipulation of physical quantities with the goal of finding a *symbolic* solution. The calculation of a *numerical* solution follows as the final step.
- We take care to ensure both *dimensional* and *mathematical* consistency at every step.
- We are disciplined in our use of *significant figures*, neither throwing away precision unnecessarily nor attributing more precision to a value than it deserves.
- We write up the solution in a logical, sequential fashion, making sure that the reader can easily see what steps we are taking even if the details aren't spelled out. We don't need to include every step of an integration, for example, as long as the starting and ending points are clear and the intermediate algebra is obvious.

The last of these requires no explanation, so we will focus in the following on the first three.

B.3.1 Symbolic solutions

First and foremost, we want to stay away from *numbers* as long as possible and analyze our problem instead in terms of *variables* (or *parameters*). *Numbers* have meanings that are tied to a specific instance of a problem. *Variables* have meanings that are independent of a specific case. If the physics is the same, then the algebra of computing the mass of the atmosphere of Mars is identical to that for Earth, only the *values* of the parameters R_E, g, and p_0 change. Therefore, it makes sense to solve the problem in general form first before substituting specific values.

We do this by letting *symbols* (often, but by no means always, Roman or Greek letters, sometimes with subscripts, superscripts, or other embellishments) represent all physical parameters, as well as any physical and mathematical constants. Your problem solution is then obtained by manipulating your chosen symbols algebraically until you have found a *self-contained* expression (or small set of related expressions) for the quantity you seek.

There are four main reasons why symbolic solutions, where possible, are preferable to strictly numerical solutions:

1. *They convey physical insight.* You can see at a glance which variables are important (or even present) and how they are related to the desired quantity.

2. You can readily verify *dimensional consistency*, which is essential for any physically valid solution (see Section B.3.2) without working through a mass of unit conversions.

3. *They are easy to validate and debug.* An instructor looking at a string of your numerical calculations cannot immediately discern whether your pathway basically leads to the desired result (especially if there is an error buried in there somewhere). A correct symbolic solution, on the other hand, will look about the same for everyone. If something is wrong, it will usually be very easy to spot what it is.

4. *They are reusable.* If you are asked to solve the same problem for several different sets of conditions, you don't need to rederive the complete solution for each case; you simply plug the new values into your symbolic solution and crunch them through your calculator. You can even code up the expressions as a Fortran, Matlab, IDL, or C++ routines so that you don't need to pick up your calculator at all. And if you do the latter, it becomes a trivial matter to generate the data needed to plot a graph of the output as a function of one or more of the inputs!

B.3.2 Dimensional consistency

It is impossible to overstate the importance of dimensional literacy[2] in physical problem solving. You *cannot* add a length to a mass. You

[2] If you need to review the fundamental physical dimensions and associated units likely to be encountered in a course like this one, see Appendix D.

cannot take a cosine or exponential of a time or of any other quantity with physical dimensions. Any problem solution that violates these and similar precepts *cannot possibly be correct!* Many of the errors I find in homework solutions could have been easily spotted, and presumably fixed, by the student if they had simply checked for dimensional consistency. This section is a brief summary of the relevant rules.

In any valid equation, the physical dimensions of both sides of the equal sign *must* be the same. For example, if A in the equation

$$A = B \cdot C$$

has dimensions of pressure, then $B \cdot C$ must also have dimensions of pressure. If

$$A = B + C \quad ,$$

then A, B, and C must all have the same dimensions. If

$$A = B \exp(C) \equiv Be^C \quad or \quad A = B \ln C \quad ,$$

then in each case B must have the dimensions of A, and C must be *dimensionless.* It follows that anytime you have expressions like

$$A = \exp(C) \equiv e^C \quad or \quad A = \ln C \quad ,$$

A *must* be dimensionless.

Dimensions of derivatives and integrals are straightforward. The dimensions of

$$\frac{df(x)}{dx}$$

are the dimensions of $f(x)$ divided by the dimensions of x. Likewise, the dimensions of

$$\int_{x_1}^{x_2} f(x)\, dx$$

are the dimensions of $f(x)$ *times* the dimensions of x.

Key fact: Carefully checking for consistency of dimensions in your computations will greatly reduce the chance for errors. If they don't conform to the above rules, then your calculation is wrong, period!

The flip side of this principle is that examining the dimensions of the given variables in a problem will often give you a clue as to how to combine them in order to obtain an expression for the quantity of interest.

Problem B.2: In the following, give the fundamental dimensions of the indicated quantity. Be sure to express your answer in terms of the *base* dimensions of "length," "mass," and/or "time."

a) $a = bC$, where a has dimensions of force and b has dimensions of mass. What are the dimensions of C ?

b) $d = F \sin(2\pi M/H)$, where d has dimensions of pressure and M has dimensions of length. What are the dimensions of H ? What are the dimensions of F ?

c) $U = (1 + aX)V$, where V has units of degrees Kelvin and X is dimensionless. What are the dimensions of a and U ?

d) If y has units of pascals and x has units of meters, what are the physical dimensions of $Z = \frac{dy}{dx}$?

e) What are the dimensions of $\int f(x)\ dx$, if $f(x)$ has dimensions of acceleration and x has dimensions of time?

f) In your own words, explain the difference between the *dimensions* of a physical quantity and its *units*.

Apparent Exceptions

Occasionally a formula may be presented to you that *seems* to require some bending of the above rules. For example, take the radar reflectivity equation $Z = AR^b$, where both Z and R have specific meteorological units (mm^6/m^3 and mm/hr, respectively). If b is a non-integer, then it becomes awkward to make any meaningful statement about the dimensions of the coefficient A. The situation becomes even worse when we take logarithms of both sides to get an expression like this:

$$\ln Z = \ln A + b \ln R \quad ,$$

in which case Z, A, and R are not allowed to have any dimensions at all (i.e., they should be "pure" numbers)! The way around this paradox is to think of Z and R not as dimensioned quantities but rather as non-dimensional *ratios* expressing the magnitude of the radar reflectivity factor and rain rate *relative to their respective standard units*.

Dimensions versus units

While it is not physically meaningful to add a length to a mass, it is certainly meaningful to add an inch to a mile. In other words, the requirement for *dimensional* consistency does not necessarily imply that all of your variables have to be specified using the same system of *units*. For example, it is perfectly acceptable to pose the following:

$$1 \text{ inch} + 1 \text{ mile} = ?$$

To find the sum, you must of course choose a single unit of length to represent your answer, and you must then express both of the lengths on the left-hand side in terms of that unit. All of the following solutions are equally valid

$$1.57\overline{82} \times 10^{-5} \text{ mile} + 1 \text{ mile} = 1.000015\overline{782} \text{ mile}$$

$$1 \text{ inch} + 63360 \text{ inch} = 63361 \text{ inch}$$

$$0.0254 \text{ meter} + 1609.344 \text{ meter} = 1609.3694 \text{ meter}$$

Of course, we have assumed above that we are speaking of adding *exactly* one inch to *exactly* one mile. After you read the section on significant figures, you will understand that if "1 mile" is taken as a *measured* quantity with only one significant figure of precision, then

$$1 \text{ inch} + 1 \text{ mile} = 1 \text{ mile} \quad !$$

B.3.3 Mathematical consistency

Quite often in atmospheric physics, we encounter first-order differential equations of the form

$$\frac{dy}{dx} = f(x,y) \quad .$$

In words, this equation tells us the following: Given particular values of x and y, an infinitesimal change dx in x will lead to an infinitesimal change dy in y; furthermore, the *ratio* of the change dy to the change dx (i.e., the left-hand side) is given by the function $f(x,y)$. In any real problem, the function $f(x,y)$ would be known, and your job would be to find the function $y(x)$—or perhaps $x(y)$—that satisfies the equation.

The standard method of solution is to try to separate variables in the following fashion

$$g(x)\ dx = h(y)\ dy,$$

where

$$f(x,y) \equiv \frac{g(x)}{h(y)},$$

Quantities like dx, dy, and $d\phi$ are known in calculus as *differentials*. It is essential to understand that they represent arbitrary *infinitesimal* (i.e., microscopic) changes in the quantities x, y, and ϕ. These have no specific value on their own and are meaningful only when compared with other differentials.

Furthermore, a differential quantity times anything is still a differential. The entire left- and right-hand sides of (B.3.3) are therefore each differentials in their own right. If we want, we can give them new names, such as

$$dU \equiv g(x)\ dx \qquad , \qquad dV \equiv h(y)\ dy \quad .$$

In meteorology, we are not normally interested in microscopic changes except as a starting point for our analysis; we want to see what happens in response to a *macroscopic* (finite) change in a variable. The process by which we get from infinitesimal to finite changes is via *integration*. In fact, integration can be thought of as the process of summing up the results of an infinite series of infinitely small steps.

For the separated differential equation given above, we normally want to integrate both sides between some initial state a and some ending state b :

$$\int_{x(a)}^{x(b)} g(x)\, dx = \int_{y(a)}^{y(b)} h(y)\, dy.$$

With luck, we will be able to find closed-form expressions for both sides that don't involve integrals.

The above procedure is straightforward and should be very familiar to you by the time you finish this course. In the meantime, you should carefully pay attention to the following essential rules:

- The *only* valid way to get from differential to macroscopic changes in a variable is to integrate. If any of your differentials mysteriously disappear without the help of a clearly indicated integration, something is wrong!

- If you integrate one side of an equation, you *must* integrate the other side, as well. You should never, *ever* wind up with an equality between a differential quantity on one side and a finite quantity on the other.

- When solving physical problems, your integrals will *always* have explicit lower and upper limits of integration (i.e., no indefinite integrals). These limits represent the starting and ending points of the physical process you are evaluating. If you leave the limits of integration off, then your solution is not mathematically correct and complete.

- The lower limits of integration on each side of the equation *must* be consistent with each other. For example, if your integral on the right-hand side is over time t, and the lower limit of integration is $t = 0$, then the lower limit of your integral over (e.g.) y on the left-hand side must represent the value of y evaluate at time $t = 0$. The same holds true for upper limits.

- When integrating over a particular variable (e.g., t), all other variables (e.g., y) on that side of the equation whose value changes with t must be *inside* that integral. This is true even if y is not already written out as an explicit function of t. If it is not, then you need to figure out how to express it as a function of t so that you can properly evaluate the integral. If you can't, then at least consider the possibility that y might more naturally belong inside the integral on the other side of the equation.

B.3.4 Numerical precision

Exact versus measured values

In the sciences, we distinguish between exact values and measured (or estimated) values. Exact values include theoretically derived quantities, like π and the ratio 4/3, both of which appear in the formula for the volume of a sphere. Also, an *assumed* (not measured) value for a physical variable may also be treated as exact, in the sense that we can say, "Consider a balloon that ascends to the 500 hPa level..." without anybody worrying about whether we really mean 500.00000 hPa or rather 500.00001 hPa.

Measured values, on the other hand, are inherently approximate. When we read an aneroid barometer, we can be pretty sure of getting a reading that is correct to the nearest hPa. With practice, we can go a step further and *estimate* to the nearest 0.1 hPa. But we can say nothing whatsoever about the correct value of the hundredths place. Our measurement, like all measurements, therefore has a finite *precision*.

Measures of precision

We can express the presumed precision in any value in either of two ways: via the *number of significant figures* and/or the *least significant decimal*.

Significant figure: Any digit that is *not* a leading zero. A number should be rounded or padded with trailing zeros to give it the correct number of significant figures in light of its actual precision.

Least significant decimal: The decimal place that holds the last significant digit. Decimal places are numbered starting at zero for the ones place and increasing to the left. Decimal places to the right of the decimal point are negative.[3]

Key fact: If the least significant decimal is greater than zero, then scientific notation should be used to prevent trailing zeros from being misinterpreted as significant.

Here are some examples of numerical values and their associated precision:

Example	No. Sig. Figs.	Least Sig. Decim.
1.37	3	-2
54.389	5	-3
1010.	4	0
1.01×10^3	3	1
0.000002	1	-6
15.0 million	3	5

The number of significant figures effectively tells you the *relative* (or fractional) uncertainty in the value. For example, a value with three significant figures is believed to be known to within about one part in 10^2–10^3 (depending on the exact value).

The least significant decimal, on the other hand, is a measure of the *absolute* uncertainty. For example, if the least significant decimal is -2, then we believe we know the value to approximately the nearest 0.01, regardless of whether the number has five significant figures (e.g., 529.03) or only one (e.g., 0.05).

[3]In other words, decimal places are numbered according to the base-10 logarithm of the value of the place.

Key fact: When *multiplying* or *dividing* two values, the number of significant figures of your result is the *lesser* of the number of significant figures of your two values.

For example, if x has five significant figures and y has only two, then the values of both xy and x/y have only two significant figures.

Key fact: When *adding* or *subtracting* two values, the least significant decimal of your result is the *greater* of the least significant decimals of your two values.

For example, if x is significant to the tenths place and y is significant to the thousandths place, then $x + y$ or $x - y$ is significant only to the tenths place.

B.3.5 Summary

Let us now quickly recap our guidelines for writing up problem solutions:

1. *Identify* the relevant variables in the problem and *assign symbols* to represent those variables. Also, determine the *physical dimensions* of each variable.

2. Ignoring the *values* of the variables for now, *solve* the given problem *symbolically*. That is, express the answer to the problem mathematically in terms of the symbols representing your known variables (and perhaps any new variables you choose to define).

3. *Verify* that your symbolic solution is dimensionally consistent—that is, that the dimensions of the right-hand side are the same as the left-hand side. If they are inconsistent, you are *guaranteed* to have made an error!

4. *Convert* the numerical values of all given variables to a *consistent set of units*. This will normally be SI. Thus, every variable whose dimensions include a length (for example) should have that length expressed in units of meters.

5. Then—and *only* then—plug the above numerical values into your symbolic solution and crank them through your calculator. There is no need to mess around with units at this point, because if the variables all had consistent units going into the calculator, the numerical result that appears on your calculator *must* be in terms of the same set of units.

6. If required, you may now convert the units of the answer to the final units requested in the problem. Whether you have to do this or not, *ensure that the actual units of your final answer are clearly displayed.* Unless the quantity you have solved for is inherently nondimensional, a numerical value with no units is about as useful as a book with no words.

7. *Persuade yourself* that the numerical result you found makes sense! For example, if you calculate a raindrop radius of 10 meters as the answer to a certain homework problem, you should immediately recognize that something is wrong, either with the problem itself, or (more likely) with your solution. Also check the *sign* of your result and see whether it is consistent with your expectations—probably the majority of computational errors I see involve sign errors.

It is important to recognize that, in the above procedure, your calculator does not come into play at all until step 4. If you have to solve the same problem a second time, say with a new set of values for your given variables, you don't need to repeat the entire derivation; you only need to jump right back to step 4. *Step 2 is the one responsible for most of the thinking work on most homework and test problems.*

B.3.6 A good example

Problem: Estimate the *total mass* (in kg) of the atmosphere M_{tot}, using the following information: Acceleration due to gravity g = 32.17 ft/s^2

Radius of the Earth R_E = 3960 miles
Average sea level pressure p_0 = 1013.2 millibars
Conversions: 1 m = 3.28 ft 1 mile = 1609.756 m

Reasoning:

1. pressure $p_0 = \frac{\text{force}}{\text{area}} = \frac{\text{mass}\times g}{\text{area}} \quad \rightarrow \quad \frac{\text{mass}}{\text{area}} = \frac{p_0}{g}$

2. total mass of atmosphere $= \frac{\text{mass}}{\text{area}} \times [\text{surface area of Earth}] = \frac{p_0}{g}\left[4\pi R^2\right]$

Symbolic solution:

$$\boxed{M_{\text{tot}} = \frac{p_0 4\pi R^2}{g}}$$

This is the most important part of your solution! If you get this part right, most of your "thinking" work is done. This form does not depend on the actual values of, or choice of units for, the parameters of the problem. You could repeat the calculations below for a different planet without rederiving the above equation!

Next, verify dimensional correctness. The above expression should have dimensions of mass (kg in SI units). If it does not, go back and find the error!

$$\frac{\text{Pa}\cdot\text{m}^2}{\text{m/s}^2} = \frac{\text{N}\cdot\text{s}^2}{\text{m}} = \frac{\text{kg}\cdot(\text{m/s}^2)\cdot\text{s}^2}{\text{m}} = \text{kg} \quad \surd$$

Finally, substitute numerical values of parameters **after** converting to consistent set of units (in this case SI):

$g = 32.17\ \frac{\text{ft}}{\text{s}^2}\cdot\frac{1\text{ m}}{3.28\text{ ft}} \quad \rightarrow \quad g = 9.808\ \frac{\text{m}}{\text{s}^2}$

$R = 3960\text{ mi}\cdot\frac{1609.8\text{ m}}{\text{mi}} \quad \rightarrow \quad R = 6.374\times10^6\text{ m}$

$p_0 = 1013.2\text{ mb}\cdot\frac{100\text{ Pa}}{\text{mb}} \quad \rightarrow \quad p_0 = 1.013\times10^5\text{ Pa}$

Numerical solution: $M_{\text{tot}} = \frac{(1.013\times10^5)4\pi(6.374\times10^6)^2}{9.808} = 5.27\times10^{18}\text{ kg}$

1. Note that we didn't need to explicitly compute the units in the final calculation because we had already verified that the *symbolic* solution correctly yielded dimensions of mass. If we then specify the values of all parameters in SI units, the results must also be in SI units!

2. For most problems in this book, three or four significant figures in the final numerical solution is sufficient precision. In this case, we chose three. We therefore don't really need to retain more than four significant figures (one more than the desired precision of our final result) for the value of any parameter going into the problem.

B.4 Checklist for Homework Solutions

- ☐ The symbolic part of the solution *must* take the form of a self-contained formula, or small set of formulas.
- ☐ A box should be drawn around the final solutions (both symbolic and numerical), so that they can be clearly identified when grading.
- ☐ All symbols appearing in the formula(s) *must* represent either (a) parameters given in the original problem, (b) quantities defined inside the solution box, or (c) quantities for which a symbolic expression was provided in a previous part of the same homework problem.
- ☐ Each solution *must* be dimensionally consistent (see Section B.3.2) to receive even partial credit.[4]
- ☐ If the starting point for the solution involves differentials, make certain that any integration is shown explicitly on both sides of the equation, with appropriate (and consistent) limits.
- ☐ There should never appear any *numerical values* in the symbolic formula that represent physical (as opposed to purely mathematical) quantities. Any such value should be replaced with an appropriately defined symbol or combination of symbols.
- ☐ There should never appear any numerical values that are decimal approximations to an exact number. For example, π should always appear in a formula as that symbol, *not* as 3.1416. Even for rational numbers, decimal representations are best avoided: use $5x/4y$ rather than $1.25x/y$.
- ☐ If a physical calculation involves temperature, there's a high likelihood that the temperature should be expressed as an *absolute* temperature. *Verify* that you have converted to Kelvin wherever required. Note: for temperature *differences* it does not matter whether you use Celsius or Kelvin—the *size* of one degree is the same for both.
- ☐ Derived or measured numerical values required as inputs to calculations should normally preserve at least four significant figures of precision, where possible—more if needed to satisfy the requirement below. Assumed values should be treated as exact.
- ☐ Where possible, final numerical values for your solution should be given with a precision of three significant figures, unless otherwise noted. However, the stated precision of the result should not be greater than that justified by the lowest-precision input to the calculation. Possible exceptions: temperatures should normally be given to the nearest tenth of a degree, pressures should be given to *at least* the nearest hectopascal, even when this means four significant figures. Temperature *differences* should be calculated to the nearest tenth of a degree, or to three significant figures, whichever is the greater precision.

[4]This may sound draconian, but checking for dimensional consistency is so easy, and so important, that there is never a valid reason to skip this step.